GEOMETRY OF QUANTUM COMPUTATION

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Preface

The book is concerned with the geometric theory of computation. We have in mind the known paradigm according to which the computation is the combination of an algorithm and a computation process. The latter process is performed by some dynamical system. If for the description of the computing dynamical system a configuration space with non-trivial geometry or topology is chosen, one can talk about the geometric character of the computation. The book examines the known holonomic and topological models of quantum computation and the new monodromic model of quantum computation is discussed in detail.

Monodromic quantum computing is a geometric model of quantum computations which is based on the idea developed in the work of the author of the book, according to which the elementary gates necessary for performing quantum computations may be realized using monodromy matrices induced by systems of differential equations of Fuchs type. As it is known, in the holonomic model of quantum computation (holonomic quantum computing) the space of quantum bits (qubits) is a finite dimensional vector space over the field of complex numbers, while the holonomy provides transformation of qubits from one state to another. On the other hand, the topological model of quantum computations is based on finding such representations of braid groups in the group of nondegenerate complex matrices whose images would form everywhere dense subgroups.

In the monodromic quantum computations, the fibre of the vector bundle induced by a Fuchs system is considered as the space of qubit states, whose dynamics is realized by the monodromy operators. From the analytic theory of differential equations, a necessary and sufficient condition for obtaining a universal system of quantum gates is known; solvability of the Riemann-Hilbert monodromy problem ensures existence of a Fuchs system corresponding to an arbitrary system of quantum gates.

Applications of the analytic theory of differential equations and geometric control theory in various models of quantum physics has a short history and today to be a lot of promising approaches are considered. Experiments show that the nanostructures do not have trivial geometry, so their study includes the role of geometry as an important ingredient.

The theory of classical computation was put on a firm foundation by proofs that Turing machines, possibly supplemented by random-number generation, can efficiently simulate classical systems, and, in particular, all other proposed models of universal classical computation. Thus, computational complexity results obtained within the Turing machine model, or any of the equivalent models, are universally applicable, up to polynomial factors. The field of quantum computing originated with Feynman’s observation that quantum-mechanical systems represent an apparent exception to this efficient mutual simulatability:
simulating quantum-mechanical systems with \( n \) degrees of freedom on a classical computer requires time scaling exponentially in \( n \). Feynman conjectured that this exception was an indication of two classes of computation: classical and quantum, with all quantum models able to simulate each other efficiently, and all classical models able to simulate each other, but not quantum models, efficiently. Deutsch defined quantum Turing machines and proposed them as a concrete model of a universal quantum computer able efficiently to simulate all others. Subsequently proposed models of quantum computation, such as the quantum circuit model, continuous and discrete quantum walks, topological quantum computers, and adiabatic quantum computers, have all been proven to be polynomially equivalent to the quantum Turing machine model. Today, the model of quantum computation most widely used by theorists is the quantum circuit model. Feynman’s original question about whether a universal quantum computer can efficiently simulate not only other models of quantum computation but also naturally occurring quantum systems has also been addressed: quantum circuits have been designed to simulate the evolution for time \( t \) of systems of \( n \) nonrelativistic particles using \( \text{poly}(n,t) \) gates.

The First chapter of the book contains all necessary definitions and basic knowledge about general theory of quantum computation. In chapter 2 a time-dependent periodic Hamiltonian admitting exact solutions is applied to construct a set of universal gates for quantum computation. The approach is based on transformation of soluble time-independent equations into time-dependent ones by employing a set of special time-dependent transformation operators. A class of periodic time-dependent Hamiltonians with cyclic solutions is constructed in a closed analytic form and the nonadiabatic geometric phase is determined in terms of the obtained solutions.

In chapter 3 is considered a three-level atom with energy levels \( |i\rangle, |g\rangle \) and \( |e\rangle \) in the so-called Lambda, Ladder and Vee configurations. It is assumed that the direct transition between \( |g\rangle \) and \( |e\rangle \) is forbidden and the dipole moments of the two transitions are perpendicular. The atom interacts with a two-mode quantum electromagnetic field. The coupled system is investigated using the rotating wave approximation. In each configuration is obtained a model similar to the Jaynes-Cummings model in the case of a two-level atom and the time evolution of the system in a semi-classical approximation as well as in a fully quantum description is studied.

In chapter 4 applications of geometric control theory methods on Lie groups and homogeneous spaces to the theory of quantum computations are investigated. These methods are shown to be very useful for the problem of constructing a universal set of gates for quantum computations: the well-known result that the set of all one-bit gates together with almost any one two-bit gate is universal is considered from the control theory viewpoint.

Differential geometric structures such as the principal bundle for the canonical vector bundle on a complex Grassmann manifold, the canonical connection form on this bundle, the canonical symplectic form on a complex Grassmann manifold and the corresponding dynamical systems are investigated. The Grassmann manifold is considered as an orbit of the co-adjoint action and the symplectic form is described as the restriction of the canonical Poisson structure on a Lie coalgebra. The holonomy of the connection on the principal bundle over Grassmannian and its relation with Berry phase is considered and investigated for the trajectories of Hamiltonian dynamical systems.

In chapter 5 is considered a model of the quantum computation, based on the mon-
odromy representation of a Fuchsian system. The rôle of local and entangling operators in monodromic quantum computing is played by monodromy matrices of connections with logarithmic singularities acting on the fiber of a holomorphic vector bundle as on the space of qubits. The leading theme is the problem of construction of a set of universal gates as monodromy operators induced from a connection with logarithmic singularity. In the formal scheme developed in this chapter can be incorporated already known models-topological and holonomic. Discussed are the possible applications of $1 - D$ direct and inverse scattering problem to the design of universal quantum gates for quantum computation. The potentials generating some universal gates are described.

The book is based on materials of a research program on the theory quantum computing which has been developed for several years at the Georgian National Academy of Sciences and later jointly with the Laboratory of Information Technologies at the Joint Institute for Nuclear Research in Dubna, together with Z. Giunashvili, M. Jibladze, G. Khimshiashvili, Z. Melikishvili, A. Suzko and R. Tevzadze. In the book are included joint works with them. The author is grateful to them for long-standing fruitful collaboration.

With the aim of geometrizing some topics in function theory, the author frequently benefitted from advice of Prof. B. Bojarski. Long-standing scientific interaction with Prof. B. Bojarski enabled an adequate adjustment of complex analytic methods to the geometric questions of quantum computing.

The author expresses his gratitude to Prof. A. Agrachev for hosting in SISSA. His expert conversations on geometric control theory helped the author in clarifying some questions concerning control theory of quantum systems.

Approximately half of the contents of the book is available for university students specializing in mathematics, physics and computer science. In the book there are questions for highly qualified specialists, new ideas and the possible ways to their solution.
References


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Quantum Computation is a very attractive and challenging task in this century. After the breakthrough by P. Shor [1] there has been remarkable progress in Quantum Computer or Computation (QC briefly). This discovery had a great influence on scientists. This drove not only theoreticians to finding other quantum algorithms, but also experimentalists to building quantum computers. See [2] and [3], [4] in outline. On the other hand, Gauge Theories are widely recognized as the basis in quantum field theories. Therefore it is very natural to intend to include gauge theories in QC · · · a constru