Research Books:


This research treatise develops a mathematical theory of the beautiful but challenging subject of the Feynman path integral approach to quantum physics, and of the closely related topic of Feynman’s operational calculus for noncommuting operators. It was written over a period of about ten years (Dec. 1989–Dec. 1999) and provides the most complete mathematical treatment of these subjects to date.

Some advantages of the approaches to the Feynman integral which are treated in detail in this book are the following: the existence of the Feynman integral is established for very general potentials in all four cases; under more restrictive but still broad conditions, three of these Feynman integrals agree with one another and with the unitary group from the usual approach to quantum dynamics; these same three Feynman integrals possess pleasant stability properties. The background material in mathematics and physics that motivates the study of the Feynman integral and Feynman’s operational calculus is discussed, and detailed proofs are provided for the central results. The last chapter discusses topics in contemporary physics and mathematics (including knot theory and low-dimensional topology) where heuristic Feynman integrals have played a significant role.

Table of Contents of the Book [RB1]:

Preface. Acknowledgements.


**Chp. 2: The Physical Phenomenon of Brownian Motion.** 2.1: A Brief Historical Sketch. 2.2: Einstein’s Probabilistic Formula.

**Chp. 3: Wiener Measure.** 3.1: There is No Reasonable Translation Invariant Measure on Wiener Space. 3.2: Construction of Wiener measure. 3.3: Wiener’s Integration Formula and Applications. Finitely-based functions. Applications. Axiomatic description of the Wiener process. 3.4: Nondifferentiability of Wiener Paths. d-dimensional Wiener measure and Wiener process. 3.5: Appendix: Converse Measurability Results. 3.6: Appendix: B(X×Y) = B(X) ⊗ B(Y).

**Chp. 4: Scaling in Wiener Space and the Analytic Feynman Integral.** 4.1: Quadratic Variation of Wiener Paths. 4.2: Scale Change in Wiener Space. 4.3: Translation Pathologies. 4.4: Scale-Invariant Measurable Functions. 4.5: The Scalar-Valued Analytic Feynman Integral. 4.6: The Nonexistence of Feynman’s “Measure”. 4.7: Appendix: Some Useful Gaussian-Type Integrals. 4.8: Appendix: Proof of Formula (4.2.3a).

**Chp. 5: Stochastic Processes and the Wiener Process.** 5.1: Stochastic Processes and Probability Measures on Function Spaces. 5.2: The Kolmogorov Consistency Theorem. 5.3: Two Realizations of the Wiener Process.


**Chp. 8: Semigroups of Operators: An Informal Introduction.**


Chp. 15: Generalized Dyson Series, the Feynman Integral and Feynman’s Operational Calculus. 15.1: Introduction. 15.2: The Analytic Operator-Valued Feynman Integral. Notation and definitions. The analytic (in mass) operator-valued Feynman integral $K$. Preliminary results. 15.3: A Simple Generalized Dyson Series ($\eta = \mu + \omega \delta_\tau$). The Classical Dyson series. 15.4: Generalized Dyson Series: The General Case. 15.5: Disentangling via Perturbation Expansions: Examples. A single measure and potential. Several measures and potentials. 15.6: Generalized Feynman Diagrams. 15.7: Commutative Banach Algebras of Functionals. The disentangling algebras $A_\tau$. The time-reversal map on $A_\tau$ and the natural physical ordering. Connections with Feynman’s operational calculus.


solution. **Propagator and explicit solution.** 17.3: Derivation of the Integral Equation in a Simple Case \( (\eta = \mu + \omega \delta_t) \). **Sketch of the proof when \( \nu \) is finitely supported.** 17.4: Discontinuities of the Solution to the Evolution Equation. **The time discontinuities.** Differential equation and change of initial condition. 17.5: Explicit Solution and Physical Interpretations. **Continuous measure: uniqueness of the solution.** Measure with finitely supported discrete part: propagator and explicit solution. Physical interpretations in the quantum-mechanical case. Physical interpretations in the diffusion case. Further connections with Feynman’s operational calculus. 17.6: The Feynman-Kac Formula with a Lebesgue-Stieltjes Measure: The General Case (Arbitrary Measure \( \eta \)). **Integral equation (integrated form of the evolution equation).** Basic properties of the solution to the integral equation. Quantum-mechanical case: reformulation in the interaction (or Dirac) picture. Product integral representation of the solution. Distributional differential equation (true differential form of the evolution equation). Unitary propagators. Scattering matrix and improper product integral. **Sketch of the proof of the integral equation.**

**Chp. 18: Noncommutative Operations on Wiener Functionals, Disentangling Algebras and Feynman’s Operational Calculus.** 18.1: Introduction. 18.2: Preliminaries: Maps, Measures and Measurability. 18.3: The Noncommutative Operations \(*\) and \(+^o\). 18.4: The Functional Integrals \( K_\lambda \) and the Operations \(*\) and \(+^o\). 18.5: The Disentangling Algebras \( \Lambda_\alpha \), the Operations \(*\) and \(+^o\), and the Disentangling Process. **Examples:** trigonometric, binomial and exponential formulas. 18.6: Appendix: Quantization, Axiomatic Feynman’s Operational Calculus, and Generalized Functional Integral. Algebraic and analytic axioms. Consequences of the axioms. **Examples:** the disentangling algebras and analytic Feynman integrals.

**Chp. 19: Feynman’s Operational Calculus and Evolution Equations.** 19.1: Introduction and Hypotheses. Feynman’s operational calculus as a generalized path integral. Exponentials of sums of noncommuting operators. Disentangling exponentials of sums via perturbation series. Local and nonlocal potentials. **Hypotheses.** 19.2: Disentangling \( \exp[-\tau\alpha + \int_0^t \beta(s) \mu(ds)] \). 19.3: Disentangling \( \exp[-\tau\alpha + \int_0^t \beta_1(s) \mu_1(ds) + \ldots + \int_0^t \beta_n(s) \mu_n(ds)] \). 19.4: Convergence of the Disentangled Series. 19.5: The Evolution Equation. 19.6: Uniqueness of the Solution to the Evolution Equation. 19.7: Further Examples of the Disentangling Process. **Nonlocal potentials relevant to phenomenological nuclear theory.**

**Chp. 20: Further Work on or Related to the Feynman Integral.** 20.1: Transform Approaches to the Feynman Integral. References to Further Approaches. **A. The Fresnel Integral and Other Transform Approaches to the Feynman Integral.** The Fresnel integral. Properties of the Fresnel integral. An approach to the Feynman integral via the Fresnel integral. Advantages and disadvantages of Fresnel integral approaches to the Feynman integral. The Feynman map. The Poisson process and transforms. A “Fresnel integral” on classical Wiener space. The Banach algebras \( \Sigma \) and
\( \Phi H \) are the same. Consequences of the close relationship between \( \Sigma \) and \( \Phi H \). More Functions in \( \Phi H \). A unified theory of Fresnel integrals: introductory remarks. Background material. A unified theory of Fresnel integrals (continued). The Fresnel classes along with quadratic forms. The classes \( \Gamma^q(H) \) and \( \Gamma^q(B) \). Quadratic forms extended. Functions in the Fresnel class of an abstract Wiener space: examples of abstract Wiener spaces. Fourier-Feynman transforms, convolution, and the first variation for functions in \( \Sigma \).

**B. References to Further Approaches to the Feynman Integral.**


**A. Knot Invariants and Low-Dimensional Topology.**

The Jones polynomial invariant for knots and links. Witten’s topological invariants via Feynman path integrals. Further developments: Vassiliev invariants and the Kontsevich integral.

**B. Further Comments and References on Subjects Related to the Feynman Integral.**


**References.**

**Index of Symbols. Author Index. Subject Index.**

---------------------------------------------------


[This refereed research monograph consists entirely of new research carried out by the authors over a period of about five years (March 1995—Nov. 1999). It develops a mathematical theory of ‘complex dimensions’ of fractal strings, and of the oscillations intrinsic to the geometry and the spectrum of the associated fractals. In particular, new results about the critical zeros of zeta functions are established and a geometric reformulation of the (Extended) Riemann Hypothesis is obtained in terms of the notion of complex dimension and of the frequency spectrum of fractal strings. On the fractal side, for example, precise explicit formulas are obtained for the volume of tubular neighborhoods of self-similar (and other fractal) strings.

In the long-term, this work is aimed at putting fractal geometry in an arithmetic context and, conversely, at putting various aspects of number theory (such as the theory of... ]
Dirichlet series, for example) in a geometric framework.

--------------------------------------------------


This is a sequel to and a greatly expanded version of the theory of complex fractal dimensions first developed in [RB2]. It contains a large amount of new research material (including several new chapters, sections, and appendices, along with many new examples and applications), almost all of which is directly connected to the work of the authors (and their collaborators) since the publication of [RB2], and a portion of which appeared in print for the first time in the present book.

Overview (of the approx. 200 pages of new material): New chapters on ‘self-similar flows’ (Chp. 7) and on ‘quasiperiodic patterns of self-similar strings’ (Chp. 3), providing a much more precise understanding (than in [RB2]) of the complex dimensions of (nonlattice) self-similar fractals (in R) and dynamical systems, as well as of the error terms in the associated ‘explicit formulas’. A new geometric description of self-similar fractal strings (in §2.1.1) and a discussion of self-similar strings with multiple generators (in Chp. 2 and throughout the rest of the book). New (and previously entirely unpublished) section (§6.3.3) on an (operator-valued) Euler product attached to the spectrum of a fractal string (generalizing that for the Riemann zeta function ζ(s) but also converging in the critical strip 0 < Re s < 1); study of the eigenvalue spectrum of the corresponding ‘spectral operator’.

New pointwise tube formulas (§8.1.1), with a number of new applications to tube formulas for self-similar strings (in §8.4); new results on average Minkowski content (§8.4.3) and on error terms for nonlattice strings (§8.4.4). New sections (§11.1.1 and §11.4.1) on recent results on zeros of zeta functions in finite arithmetic progressions (extending those of the authors in §9.2 and §9.3 of Chapter 9 of [RB2]). Discussion in the last chapter (Chp. 12) of a number of new topics, including recent results on random fractal strings, quantized strings (fractal membranes), and especially on the complex dimensions of the Koch snowflake curve and its generalizations (via an associated ‘tube formula’), as well as of an outline of a higher-dimensional theory of complex dimensions of self-similar systems and fractals. Also, further discussion of a possible cohomological interpretation of the complex dimensions. New appendix (App. A) on Nevanlinna theory and its applications in this context, and a new section (§B.4 of App. B) on ‘two-variable zeta functions’. New examples, illustrations, theorems, and proofs, scattered throughout the book.]

Table of Contents of the Book [RB3]:

- 7 -
Preface. List of Figures. List of Tables.

Overview. Introduction.


Chp. 2: Complex Dimensions of Self-Similar Fractal Strings. 2.1: Construction of a Self-Similar Fractal String. 2.1.1: Relation with self-similar sets. 2.2: The Geometric Zeta Function of a Self-Similar String. 2.2.1: Self-similar strings with a single gap. 2.3: Examples of Complex Dimensions of Self-Similar Strings. 2.3.1: The Cantor string. 2.3.2: The Fibonacci string. 2.3.3: The modified Cantor and Fibonacci strings. 2.3.4: A string with multiple poles. 2.3.5: Two nonlattice examples: the two-three string and the golden string. 2.4: The Lattice and Nonlattice Case. 2.5: The Structure of the Complex Dimensions. 2.6: The Asymptotic Density of the Poles in the Nonlattice Case. 2.7: Notes.

Chp. 3: Complex Dimensions of Nonlattice Self-Similar Strings: Quasiperiodic Patterns and Diophantine Approximation. 3.1: Dirichlet Polynomial Equations. 3.1.1: The generic nonlattice case. 3.2: Examples of Dirichlet Polynomial Equations. 3.2.1: Generic and nongeneric nonlattice equations. 3.2.2: The complex roots of the golden plus equation. 3.3: The Structure of the Complex Roots. 3.4: Approximation of a Nonlattice Equation by Lattice Equations. 3.4.1: Diophantine approximation. 3.4.2: The quasiperiodic pattern of the complex dimensions. 3.4.3: Applications to nonlattice strings. 3.5: Complex Roots of a Nonlattice Dirichlet Polynomial. 3.5.1: Continued fractions. 3.5.2: Two generators. 3.5.3: More than two generators. 3.6: Dimension-Free Regions. 3.7: The Dimensions of Fractality of a Nonlattice String. 3.7.1: The density of the real parts. 3.8: A Note on the Computations.

Chp. 4: Generalized Fractal Strings Viewed as Measures. 4.1: Generalized Fractal Strings. 4.1.1: Examples of generalized fractal strings. 4.2: The Frequencies of a Generalized Fractal String. 4.2.1: Completion of the harmonic string: Euler product. 4.3: Generalized Fractal Sprays. 4.4: The Measure of a Self-Similar String. 4.4.1: Measures with a self-similarity property. 4.5: Notes.

Chp. 5: Explicit Formulas for Generalized Fractal Strings. 5.1: Introduction. 5.1.1: Outline of the proof. 5.1.2: Examples. 5.2: Preliminaries: The Heaviside Function. 5.3: Pointwise Explicit Formulas. 5.3.1: The order of the sum over the complex dimensions. 5.4: Distributional Explicit Formulas. 5.4.1: Extension to more general test functions. 5.4.2: The order of the distributional error term. 5.5: Example: The Prime Number Theorem. 5.5.1: The Riemann-von Mangoldt formula. 5.6: Notes.

6.1.3: The Weyl term. 6.1.4: The distribution \( x^\omega (\log x)^\mu \). 6.2: Explicit Formulas for Lengths and Frequencies. 6.2.1: The geometric counting function of a fractal string. 6.2.2: The spectral counting function of a fractal string. 6.2.3: The geometric and spectral partition functions. 6.3: The Direct Spectral Problem for Fractal Strings. 6.3.1: The density of geometric and spectral states. 6.3.2: The spectral operator and its Euler product. 6.4: Self-Similar Strings. 6.4.1: Lattice strings. 6.4.2: Nonlattice strings. 6.4.3: The spectrum of a self-similar string. 6.5: Examples of Non-Self-Similar Strings. 6.5.1: The a-string. 6.5.2: The spectrum of the harmonic string. 6.6: Fractal Sprays. 6.6.1: The Sierpinski drum. 6.6.2: The spectrum of a self-similar spray.

Chp. 7: Periodic Orbits of Self-Similar Flows. 7.1: Suspended Flows. 7.1.1: The zeta function of a dynamical system. 7.2: Periodic Orbits, Euler Product. 7.3: Self-Similar Flows. 7.3.1: Examples of self-similar flows. 7.3.2: The lattice and nonlattice case. 7.4: The Prime Orbit Theorem for Suspended Flows. 7.4.1: The prime orbit theorem for self-similar flows. 7.4.2: Lattice flows. 7.4.3: Nonlattice flows. 7.5: The Error Term in the Nonlattice Case. 7.5.1: Two generators. 7.5.2: More than two generators. 7.6: Notes.


Chp. 10: Generalized Cantor Strings and Their Oscillations. 10.1: The Geometry of a Generalized Cantor String. 10.2: The Spectrum of a Generalized Cantor String. 10.2.1: Integral Cantor strings: \( a \)-adic analysis of the geometric and spectral oscillations. 10.2.2: Nonintegral Cantor strings: analysis of the jumps in the spectral counting function. 10.3: The Truncated Cantor String. 10.3.1: The spectrum of the truncated Cantor string. 10.4: Notes.

Chp. 11: The Critical Zeros of Zeta Functions. 11.1: The Riemann Zeta Function: No Critical Zeros in Arithmetic Progression. 11.1.1: Finite arithmetic progressions of zeros. 11.2: Extension to Other Zeta Functions. 11.3: Density of Nonzeros on Vertical Lines. 11.3.1: Almost arithmetic progressions of zeros. 11.4: Extensions to L-Series. 11.4.1: Finite arithmetic progressions of zeros of L-series. 11.5: Zeta Functions of Curves Over Finite Fields.


Bibliography. Acknowledgements.

Conventions. Index of Symbols. Author Index. Subject Index.


[The (physically motivated) theory proposed and developed in this research monograph represents approximately ten years of the author’s research on this subject (between about 1996 and 2006). It realizes a synthesis of aspects of string theory, noncommutative geometry, the author (and his collaborators)’ theory of fractal strings (now ‘quantized’ and referred to in this new form as ‘fractal membranes’) and their complex dimensions, as well as of number theory and arithmetic geometry. In particular, it builds upon and expands—but is also in many ways quite different from—the author’s earlier (joint) work in [JA24-27] or in [RB2, RB3]. Much of the material presented in the main part of this book (with
the exception of a portion of Chapter 2 and the first section of Chapter 3) is original and
published for the first time.]

Table of Contents of the Book [RB4]:

Preface. Overview.

Chp. 1: Introduction. 1.1: Arithmetic and Spacetime Geometry. 1.2.: Riemannian, Quantum
and Noncommutative Geometry. 1.3: String Theory and Spacetime Geometry. 1.4: The
Riemann Hypothesis and the Geometry of the Primes. 1.5: Objectives, Motivations and
Plan of this Book.

Chp. 2: String Theory on a Circle and T-Duality: Analogy with the Riemann Zeta
Function. 2.1: Quantum Mechanical Point-Particle on a Circle. 2.2: String Theory on a
Circle and the Existence of a Fundamental Length. 2.2.1: String theory on a circle. 2.2.2:
Circle duality (T-duality for circular spacetimes). 2.2.3: T-duality and the existence of a
fundamental length. 2.3: Noncommutative Stringy Spacetimes and T-Duality. 2.3.1: Target
space duality and noncommutative geometry. 2.3.2: Noncommutative stringy spacetimes:
Fock spaces, vertex algebras and chiral Dirac operators. 2.4: Analogy with the Riemann
Zeta Function: Functional Equation and T-Duality. 2.4.1: Key properties of the Riemann
zeta function. 2.4.2: The functional equation, T-duality and the Riemann hypothesis. 2.5:
Notes.

Chp. 3: Fractal Strings and Fractal Membranes. 3.1: Fractal Strings: Geometric Zeta
Functions, Complex Dimensions and Self-Similarity. 3.1.1: The spectrum of a fractal
string. 3.2: Fractal (and Prime) Membranes: Spectral Partition Functions and Euler
Products. 3.2.1: Prime membranes. 3.2.2: Fractal membranes and Euler products. 3.3:
Fractal Membranes vs. Self-Similarity: Self-Similar Membranes. 3.4: Notes.

Chp. 4: Noncommutative Models of Fractal Strings: Fractal Membranes and Beyond.
4.1: Connes’ Spectral Triple for Fractal Strings. 4.2: Fractal Membranes and the Second
Quantization of Fractal Strings. 4.2.1: An alternative construction of fractal membranes.
4.3: Fractal Membranes and Noncommutative Stringy Spacetimes. 4.4: Cyclic
Cohomology and a Possible Interpretation of (Dynamical) Complex Dimensions. 4.4.1:
Fractal membranes and quantum deformations: a possible connection with Haran’s real
and finite primes. 4.5: Notes.

Chp. 5: Towards an ‘Arithmetic Site’: Moduli Spaces of Fractal Strings and Membranes. 5.1: The
Set of Penrose Tilings: Quantum Space as a Quotient Space. 5.2: The Moduli Space of
Fractal Strings: A Natural Receptacle for Zeta Functions. 5.3: The Moduli Space of Fractal
Membranes: A Quantized Moduli Space of Fractal Strings. 5.4: Arithmetic Site, Frobenius
Flow and the Riemann Hypothesis. 5.4.1: The moduli space of fractal strings and
Deninger’s arithmetic site. 5.4.2: The moduli space of fractal membranes and
(noncommutative) modular flow vs. arithmetic site and Frobenius flow. 5.4.2a: Factors and their classification. 5.4.2b: Modular theory of von Neumann algebras. 5.4.2c: Reduction of type III to type II factors: automorphisms and flows. 5.4.2d: Noncommutative flows on moduli spaces and the Riemann hypothesis. 5.4.2e: Towards an extended moduli space and flow.) 5.5: Flows of Zeros and Zeta Functions: A Dynamical Interpretation of the Riemann Hypothesis. 5.5.1. Introduction. 5.5.2. Expected Properties of the Flows of Zeros and Zeta Functions. 5.5.3. Analogies with Other Geometric, Analytical or Physical Flows. (5.5.3a. Singular Potentials, Schrödinger Equation and Renormalization Flow. 5.5.3b. KMS Flow and Deformations of Pólya-Hilbert Operators. 5.5.3c. Ricci Flow and Geometric Homogenization. 5.5.3d. Noncommutative KP and Geodesic Flow.) 5.6: Notes.


Acknowledgements. Bibliography.

Conventions. Index of Symbols. Author Index. Subject Index.

-------------------------------------------------

Refereed Edited Research Books:


[This volume gathers original research contributions or survey expository articles by some of the best experts in spectral geometry, along with a few papers by promising junior investigators selected by the editors. With two exceptions, the editors have reviewed and edited each paper individually.]


[This volume gathers high quality original research contributions or survey expository articles by some of the leading experts in classical and modern analysis, working in the areas of harmonic analysis, nonlinear partial differential and mathematical physics. Each contribution in this volume has been individually refereed according to strict standards set by the American Mathematical Society.]

[This volume gathers high quality original research contributions or survey expository articles by some of the leading experts working at the interface of number theory, dynamical systems and/or spectral as well as arithmetic geometry. Each contribution in the volume has been individually refereed according to strict standards set by the American Mathematical Society.]

[EB4] & [EB5]:


Subtitle of Part 1: Analysis, Number Theory, and Dynamical Systems. (Approx. 520 pages; precisely, 508 + (xiii) pages.)

Subtitle of Part 2: Multifractals, Probability and Statistical Mechanics, Applications. (Approx. 560 pages; precisely, 546 + (xiii) pages.)

[The PSPUM Series is the most prestigious proceedings series published by the American Mathematical Society. Very few volumes are published every year (or decade). (The preparation of the present two-part volume has taken about three years.) In part for this reason, as the Managing Editor of the volume (i.e., of both Parts 1 and 2), I have devoted a great deal of attention and care to the selection of the invited contributors, the refereeing process (drawing upon more than forty expert referees), and the editing of the volume. This has been an extremely time-consuming task for me, spanning over hundreds of hours, but one that I think will be ultimately worthwhile and useful to the mathematical as well as the broader scientific community.

The goal of these two books is to give an overview of the field of fractal geometry and of its applications [within mathematics (e.g., harmonic analysis, dynamical systems, number theory, probability, and mathematical physics) as well as to the other sciences (e.g., physics, chemistry, engineering, and computer graphics)], via a careful selection of research expository articles, tutorial articles, and original research papers. It should be accessible and useful to experts and non-experts alike.]

**Books in Preparation:**

[RB5] **Research Monograph:** “Noncommutativity and Time-Ordering”. (Subtitle:
"Feynman’s Operational Calculus and Beyond”.) (With Gerald W. Johnson and Lance Nielsen.)

[TB] **Textbook:** “An Invitation to Fractal Geometry and Its Applications”. (With Dana Clahane, Erin Pearse and Robert Niemeyer.)
Research is the best part to build a person future Success. Mention below are some of the best and Affordable books:- 1. Ranjit Kumar- Research Methodology A Step-by-Step G.pdf :- Download Free pdf from this website and Purchase from This Link A2A.

Research Methodology helps you to create an understanding of what is the preferred method of study. Of all the research books I have read, this is the easiest to understand. I now feel like I know exactly what I need to do – Jonathan, Pittsburgh, USA.

Why this e-book? The Ultimate Guide to Writing a Dissertation contains step-by-step guidance derived from the experience of assisting hundreds of students who have successfully completed dissertations in business studies.